# Równanie *n*-tego rzędu

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \ldots + a_1 \dot{x} + a_0 x = b_m u^{(m)} + a_{m-1} u^{(m-1)} + \ldots + b_1 \dot{u} + b_0 u$$

## Równanie statyczne (stan ustalony, punkt równowagi)

$$a_0 x = b_0 u \qquad x = \frac{b_0 u}{a_0}$$

## Warunki początkowe (*n* warunków początkowych)

$$x^{(n)}(0) = x_{0n};\ x^{(n-1)}(0) = x_{0n-1};\ \ldots;\ \dot{x}(0) = x_{01};\ x(0) = x_0$$

Zazwyczaj: $x^{(n)}(0) = x^{(n-1)}(0) = \ldots = \dot{x}(0) = 0$

Badania analityczne (równanie charakterystyczne):

$$a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 = 0$$

# Równania stanu

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + b_{11}u_1 + b_{12}u_2 \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + b_{21}u_1 + b_{22}u_2 \\ \dot{x}_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + b_{31}u_1 + b_{32}u_2 \end{cases}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

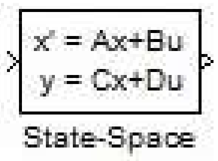
$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$ równania stanu

$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$ równania wyjściowe

Badania analityczne (równanie charakterystyczne):

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0$$

Badania symulacyjne:



# Równanie n-tego rzędu → układ n równań 1-ego rzędu

$$a_3\dddot{x}(t) + a_2\ddot{x}(t) + a_1\dot{x}(t) + a_0 x(t) = u(t)$$

$$a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

rząd ?

$$x = x_1$$

$$\dot{x}_1 = x_2 = \dot{x}$$

$$\dot{x}_2 = x_3 = \ddot{x}$$

$$\dot{x}_3 = \dddot{x}$$

$$a_3\dot{x}_3(t) + a_2 x_3(t) + a_1 x_2(t) + a_0 x_1(t) = u(t)$$

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_3(t) \\ \dot{x}_3(t) = \left[u(t) - a_0 x_1(t) - a_1 x_2(t) - a_2 x_3(t)\right]/a_3 \end{cases}$$

rząd ?

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \dfrac{-a_0}{a_3} & \dfrac{-a_1}{a_3} & \dfrac{-a_2}{a_3} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/a_3 \end{bmatrix} u(t)$$

# Równania stanu

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + b_{11}u_1 + b_{12}u_2 \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + b_{21}u_1 + b_{22}u_2 \\ \dot{x}_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + b_{31}u_1 + b_{32}u_2 \end{cases}$$

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

## Równania statyczne (stan ustalony, punkt równowagi):

$$\begin{cases} 0 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + b_{11}u_1 + b_{12}u_2 \\ 0 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + b_{21}u_1 + b_{22}u_2 \\ 0 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + b_{31}u_1 + b_{32}u_2 \end{cases}$$

$$0 = \mathbf{Ax} + \mathbf{Bu}$$

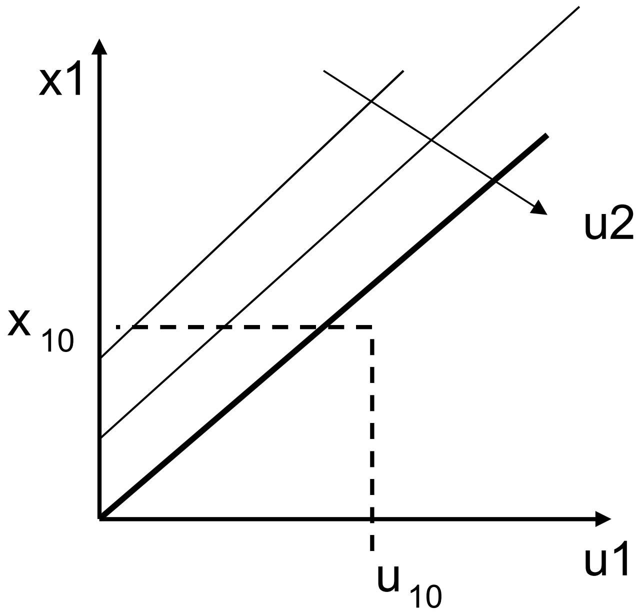
$$\mathbf{Ax} = -\mathbf{Bu}$$

$$\mathbf{x} = -\mathbf{A}^{-1}\mathbf{Bu}$$

$$x_1 = f_1(u_1, u_2)$$

$$x_2 = f_2(u_1, u_2)$$

$$x_3 = f_3(u_1, u_2)$$




charakterystyka statyczna (punkt równowagi)

## Warunki początkowe (*n* warunków)

Zazwyczaj: $\dot{x}_1(0) = 0, \dot{x}_2(0) = 0, \dot{x}_3(0) = 0$

# Równania stanu – przykład 1

$$\begin{cases} C_{vg}\dot{T}_g(t) = P_g(t) - K_g\left(T_g(t) - T_{wew}(t)\right) \\ C_{vw}\dot{T}_{wew}(t) = K_g\left(T_g(t) - T_{wew}(t)\right) - K_1\left(T_{wew}(t) - T_{zew}(t)\right) \end{cases}$$

Wejścia: $P_{g0}$, $T_{zew0}$

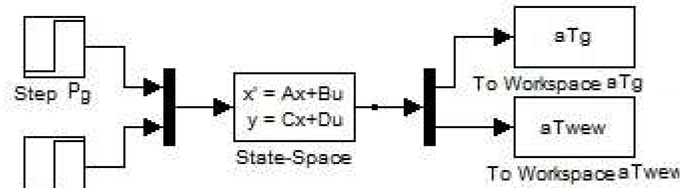
Wyjścia: $T_{g0}$, $T_{wew0}$

$$\begin{bmatrix} \dot{T}_g \\ \dot{T}_{wew} \end{bmatrix} = \begin{bmatrix} \frac{-K_g}{C_{vg}} & \frac{K_g}{C_{vg}} \\ \frac{K_g}{C_{vw}} & \frac{-K_g - K_1}{C_{vw}} \end{bmatrix} \begin{bmatrix} T_g \\ T_{wew} \end{bmatrix} + \begin{bmatrix} \frac{1}{C_{vg}} & 0 \\ 0 & \frac{K_1}{C_{vw}} \end{bmatrix} \begin{bmatrix} P_g \\ T_{zew} \end{bmatrix}$$

Wejścia: $\mathbf{u} = \begin{bmatrix} P_g \\ T_{zew} \end{bmatrix}$ Wyjścia: $\mathbf{x} = \begin{bmatrix} T_p \\ T_{wew} \end{bmatrix}$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$0 = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{x} = -\mathbf{A}^{-1}\mathbf{B}\mathbf{u}$$




**A, B, C, D, x**

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$$\begin{cases} C_{vw}\dot{T}_{wew}(t) = K_g\left(T_g(t) - T_{wew}(t)\right) - K_1\left(T_{wew}(t) - T_{zew}(t)\right) \\ C_{vg}\dot{T}_g(t) = P_g(t) - K_g\left(T_g(t) - T_{wew}(t)\right) \end{cases}$$

Wejścia: $P_{g0}$, $T_{zew0}$

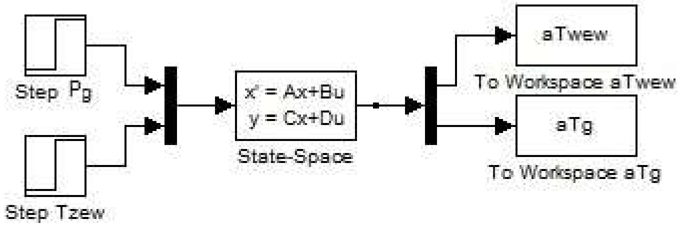
Wyjścia: $T_{wew0}$, $T_{g0}$

$$\begin{bmatrix} \dot{T}_{wew} \\ \dot{T}_g \end{bmatrix} = \begin{bmatrix} \frac{-K_g - K_1}{C_{vw}} & \frac{K_g}{C_{vw}} \\ \frac{K_g}{C_{vg}} & \frac{-K_g}{C_{vg}} \end{bmatrix} \begin{bmatrix} T_{wew} \\ T_g \end{bmatrix} + \begin{bmatrix} 0 & \frac{K_1}{C_{vw}} \\ \frac{1}{C_{vg}} & 0 \end{bmatrix} \begin{bmatrix} P_g \\ T_{zew} \end{bmatrix}$$

Wejścia: $\mathbf{u} = \begin{bmatrix} P_g \\ T_{zew} \end{bmatrix}$ Wyjścia: $\mathbf{x} = \begin{bmatrix} T_{wew} \\ T_g \end{bmatrix}$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$0 = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{x} = -\mathbf{A}^{-1}\mathbf{B}\mathbf{u}$$

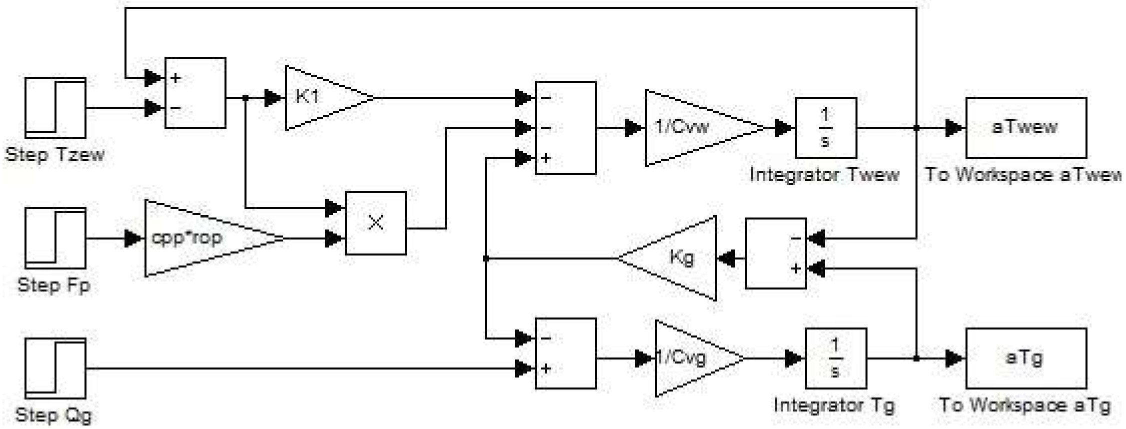


# Równania stanu – przykład 2

$$\begin{cases} C_{vw}\dot{T}_{wew}(t) = K_g\left(T_g(t)-T_{wew}(t)\right) - K_1\left(T_{wew}(t)-T_{zew}(t)\right) - c_{pp}\rho_p f_p(t)\left(T_{wew}(t)-T_{zew}(t)\right) \\ C_{vg}\dot{T}_g(t) = P_g(t) - K_g\left(T_g(t)-T_{wew}(t)\right) \end{cases},$$

Wejścia: $P_{g0}$, $T_{zew0}$, $f_{p0}$

Wyjścia: $T_{wew0}$, $T_{g0}$

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$$\begin{cases} C_{vw}\dot{T}_{wew}(t) = K_g\left(T_g(t)-T_{wew}(t)\right) - K_1\left(T_{wew}(t)-T_{zew}(t)\right) - c_{pp}\rho_p f_{p0}\left(T_{wew}(t)-T_{zew}(t)\right) \\ C_{vg}\dot{T}_g(t) = P_g(t) - K_g\left(T_g(t)-T_{wew}(t)\right) \end{cases}$$

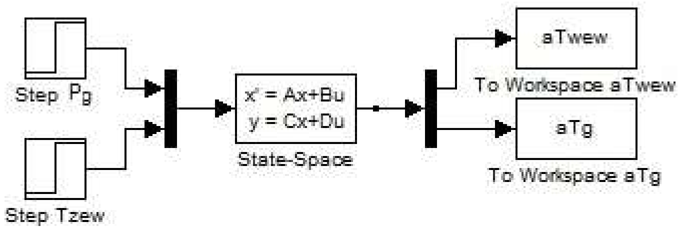
Wejścia: $P_{g0}$, $T_{zew0}$

Wyjścia: $T_{wew0}$, $T_{g0}$

$$\begin{bmatrix} \dot{T}_{wew} \\ \dot{T}_g \end{bmatrix} = \begin{bmatrix} \dfrac{-\left(K_g + K_1 + c_{pp}\rho_p f_{p0}\right)}{C_{vw}} & \dfrac{K_g}{C_{vw}} \\ \dfrac{K_g}{C_{vg}} & \dfrac{-K_g}{C_{vg}} \end{bmatrix} \begin{bmatrix} T_{wew} \\ T_g \end{bmatrix} + \begin{bmatrix} 0 & \dfrac{K_1 + c_{pp}\rho_p f_{p0}}{C_{vw}} \\ \dfrac{1}{C_{vg}} & 0 \end{bmatrix} \begin{bmatrix} P_g \\ T_{zew} \end{bmatrix}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{A} = \ldots,\ \mathbf{B} = \ldots$$




Wejścia: $\mathbf{u} = \begin{bmatrix} P_g \\ T_{zew} \end{bmatrix}$

Wyjścia: $\mathbf{x} = \begin{bmatrix} T_{wew} \\ T_g \end{bmatrix}$

$$\mathbf{x} = -\mathbf{A}^{-1}\mathbf{B}\mathbf{u}$$

# Transmitancja

*Założenie: Równanie różniczkowe liniowe*

$$a_n x^{(n)}(t) + \ldots + a_1 \dot{x}(t) + a_0 x(t) = b_m u^{(m)}(t) + \ldots + b_1 \dot{u}(t) + b_0 u(t)$$

| | | |
|---|---|---|
| $\mathcal{L}[f(t)] = f(s)$ | $\mathcal{L}[af_1(t) + bf_2(t)] = af_1(s) + bf_2(s)$ | $\mathcal{L}[\dot{f}(t)] = sf(s) - f(0_+)$ |

*Transformata funkcji* — *Twierdzenie o liniowości* — *Transformata funkcji pochodnej*

*Założenie:* $f(0_+) = 0$

$$a_n s^n X(s) + \ldots + a_1 s X(s) + a_0 X(s) = b_m s^m U(s) + \ldots + b_1 s U(s) + b_0 U(s)$$

*Równanie operatorowe*

$$\left(a_n s^n + \ldots + a_1 s + a_0\right) X(s) = \left(b_m s^m + \ldots + b_1 s + b_0\right) U(s)$$

$$\frac{X(s)}{U(s)} = \boxed{\frac{b_m s^m + \ldots + b_0}{a_n s^n + \ldots + a_0} = G(s)}$$

*Transmitancja*

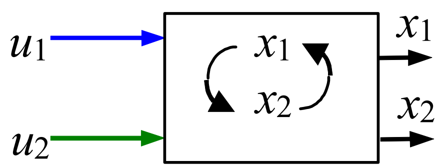
**Transmitancja** = transformata funkcji wyjściowej do transformaty funkcji wejściowej
= funkcja przejścia, opisująca sposób przetwarzania sygnału
Założenia: Równanie różniczkowe liniowe i zerowe warunki początkowe

Badania analityczne (równanie charakterystyczne):

$$M(s) = 0 \longrightarrow a_n s^n + \cdots + a_1 s + a_0 = 0$$

# Transmitancje układów wielowymiarowych

Przykład
$$\begin{cases} m_1\dot{x}_1(t) + a_1x_1(t) - a_2x_2(t) = u_1(t) \\ m_2\dot{x}_2(t) - b_1x_1(t) + b_2x_2(t) = u_2(t) \end{cases}$$

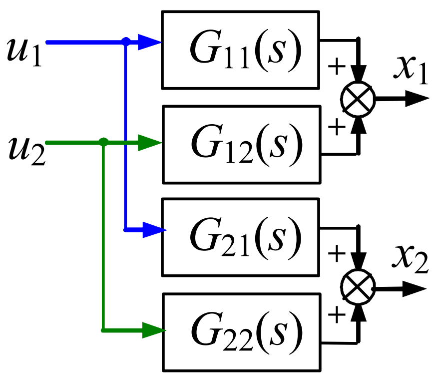


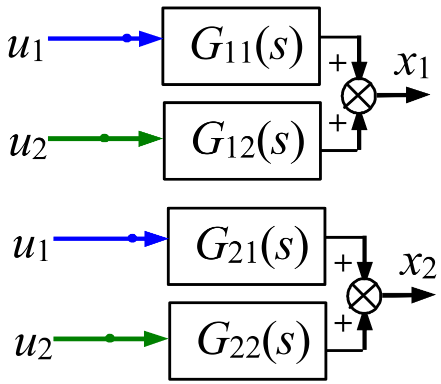

$$x_1(s) = G_{11}(s)u_1(s) + G_{12}(s)u_2(s)$$

$$x_2(s) = G_{21}(s)u_1(s) + G_{22}(s)u_2(s)$$

$$\begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$$

Pełny model



# Transmitancje układów wielowymiarowych

Przykład
$$\begin{cases} m_1\dot{x}_1(t) + a_1 x_1(t) - a_2 x_2(t) = u_1(t) \\ m_2\dot{x}_2(t) - b_1 x_1(t) + b_2 x_2(t) = u_2(t) \end{cases}$$

Operacje macierzowe

$$s\begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = \begin{bmatrix} \frac{-a_1}{m_1} & \frac{a_2}{m_1} \\ \frac{b_1}{m_2} & \frac{-b_2}{m_2} \end{bmatrix} \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} + \begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$$

$$s\mathbf{x}(s) = \mathbf{A}\mathbf{x}(s) + \mathbf{B}\mathbf{u}(s)$$

$$\begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = \begin{bmatrix} s + \frac{a_1}{m_1} & \frac{-a_2}{m_1} \\ \frac{-b_1}{m_2} & s + \frac{b_2}{m_2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$$

$$\mathbf{x}(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}\mathbf{u}(s) = \mathbf{G}(s)\mathbf{u}(s)$$

$$\begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = \begin{bmatrix} (-1)^{1+1}\frac{\det\left(\frac{m_2 s + b_2}{m_2}\right)}{\det(sI - A)} & (-1)^{2+1}\frac{\det\left(\frac{-a_2}{m_1}\right)}{\det(sI - A)} \\ (-1)^{1+2}\frac{\det\left(\frac{-b_1}{m_2}\right)}{\det(sI - A)} & (-1)^{2+2}\frac{\det\left(\frac{m_1 s + a_1}{m_1}\right)}{\det(sI - A)} \end{bmatrix} \begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$$

$$\det(sI - A) = \left(\frac{m_1 s + a_1}{m_1}\right)\left(\frac{m_2 s + b_2}{m_2}\right) - \frac{b_1 a_2}{m_1 m_2}$$

$$\begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = \begin{bmatrix} \frac{m_2 s + b_2}{M(s)} & \frac{a_2}{M(s)} \\ \frac{b_1}{M(s)} & \frac{m_1 s + a_1}{M(s)} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$$

$$M(s) = \left(m_1 s + a_1\right)\left(m_2 s + b_2\right) - b_1 a_2$$

# Transmitancje układów wielowymiarowych

Przykład
$$\begin{cases} m_1\dot{x}_1(t)+a_1x_1(t)-a_2x_2(t)=u_1(t) \\ m_2\dot{x}_2(t)-b_1x_1(t)+b_2x_2(t)=u_2(t) \end{cases}$$

Równania operatorowe
$$\begin{cases} (m_1s+a_1)x_1(s)=a_2x_2(s)+u_1(s) \\ (m_2s+b_2)x_2(s)=b_1x_1(s)+u_2(s) \end{cases}$$

Rozwiązanie
układu równań operatorowych

$$\begin{cases} M_1(s)x_1(s)=a_2x_2(s)+u_1(s) \\ M_2(s)x_2(s)=b_1x_1(s)+u_2(s) \end{cases} \longrightarrow \quad x_1(s)=\frac{a_2x_2(s)+u_1(s)}{M_1(s)}$$

$$M_2(s)x_2(s)=b_1\frac{a_2x_2(s)+u_1(s)}{M_1(s)}+u_2(s) \quad \Big|\cdot M_1(s)$$

$$M_1(s)M_2(s)x_2(s)=b_1a_2x_2(s)+b_1u_1(s)+M_1(s)u_2(s)$$

$$\boxed{x_2(s)=\frac{b_1u_1(s)+M_1(s)u_2(s)}{M_1(s)M_2(s)-b_1a_2}}$$

$$M_1(s)x_1(s)=a_2x_2(s)+u_1(s)= \quad a_2\frac{b_1u_1(s)+M_1(s)u_2(s)}{M_1(s)M_2(s)-b_1a_2}+u_1(s)$$

$$x_1(s)=\frac{a_2b_1u_1(s)+a_2M_1(s)u_2(s)+\left(M_1(s)M_2(s)-b_1a_2\right)u_1(s)}{M_1(s)\left(M_1(s)M_2(s)-b_1a_2\right)}$$

$$\boxed{x_1(s)=\frac{M_2(s)u_1(s)+a_2u_2(s)}{M_1(s)M_2(s)-b_1a_2}}$$

# Transmitancja – stan ustalony (punkt równowagi)

$$G(s)=\frac{X(s)}{U(s)} \longrightarrow X(s)=G(s)U(s)$$

$$\lim_{t\to\infty} x(t)=\lim_{s\to 0} sX(s)=\lim_{s\to 0} sG(s)U(s)$$ , jeśli istnieje $\lim_{t\to\infty} x(t)$

Dla $u(t)=1(t)$, czyli $U(s) = 1/s$

$$\lim_{s\to 0} sG(s)\frac{1}{s}=\lim_{s\to 0} G(s)$$

Dla $u(t)=k$, czyli $U(s) = k/s$

$$\lim_{s\to 0} sG(s)\frac{k}{s}=\lim_{s\to 0} G(s)k$$

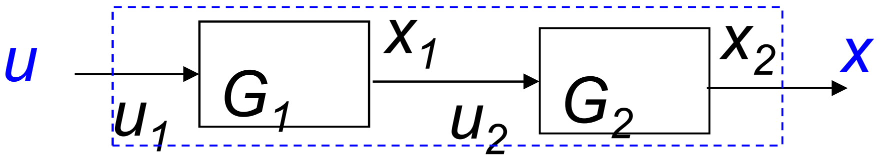
## Warunki początkowe

Z założenia zerowe warunki początkowe: $x^{(n-1)}(0)=\ldots;=\dot{x}(0)=0 \longrightarrow \mathcal{L}\left[\dot{f}(t)\right]=sf(s)-f(0_+)=sf(s)$

$$a_n x^{(n)}(t)+\ldots+a_1\dot{x}(t)+a_0x(t)=b_m u^{(m)}(t)+\ldots+b_1\dot{u}(t)+b_0u(t)$$

$$a_n s^n X(s)+\ldots+a_1 sX(s)+a_0X(s)=b_m s^m U(s)+\ldots+b_1 sU(s)+b_0U(s) \longrightarrow G(s)=\frac{X(s)}{U(s)}$$

## Transmitancja zastępcza

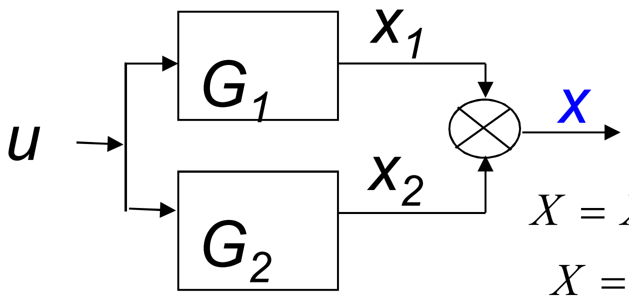



$$X = G_1 G_2 U$$

$$X_1 = G_1 U_1$$

$$X_2 = G_2 U_2 = G_2 X_1$$

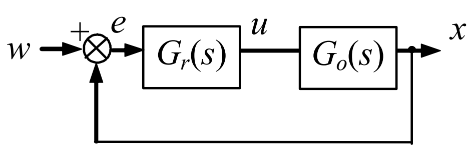
$$X_2 = G_1 G_2 U_1$$

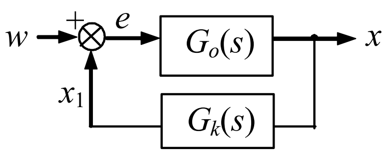



$$X = (G_1 + G_2)U$$

$$X = X_1 + X_2$$

$$X = G_1 U + G_2 U$$



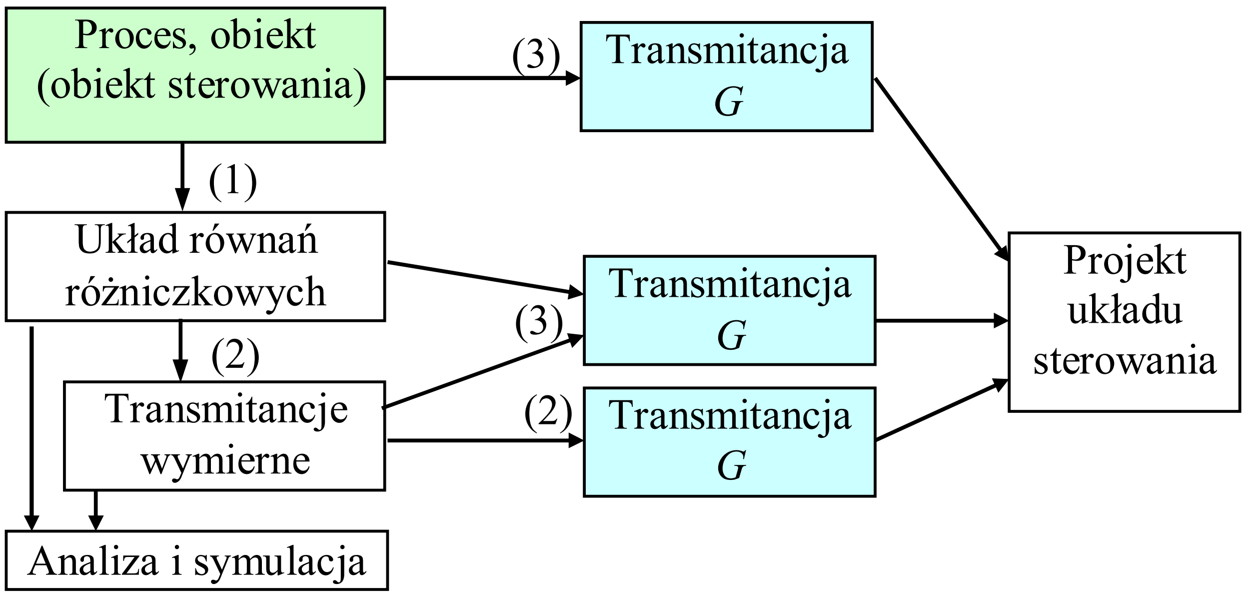

$$\begin{cases} e(s) = w(s) - x(s) \\ x(s) = G_o(s)u(s) \\ u(s) = G_r(s)e(s) \end{cases}$$

$$\frac{x(s)}{u(s)} = \frac{G_o(s)G_r(s)}{G_o(s)G_r(s) + 1}$$

$$\begin{cases} e(s) = w(s) - x_1(s) \\ x(s) = G_o(s)e(s) \\ x_1(s) = G_k(s)x(s) \end{cases}$$

$$\frac{x(s)}{u(s)} = \frac{G_o(s)}{G_o(s)G_k(s) + 1}$$

# Zastosowanie różnych typów modeli




1 – modelowanie
2 – konwersja
3 – identyfikacja

$$a_n x^{(n)}(t) + \ldots + a_1 \dot{x}(t) + a_0 x(t) = b_m u^{(m)}(t) + \ldots + b_1 \dot{u}(t) + b_0 u(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots a_{1n} \\ a_{21} & a_{22} & \ldots \quad a_{2n} \\ \ldots & & \\ a_{n1} & a_{n2} & \ldots \quad a_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_{11} & \ldots & b_{1m} \\ b_{21} & \ldots & b_{2m} \\ \ldots & & \\ b_{n1} & \ldots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \\ u_m(t) \end{bmatrix}$$

$$G(s) = \frac{b_m s^m + \ldots + b_1 s + b_0}{a_n s^n + \ldots + a_1 s + a_0}$$